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ADDING INTERIOR POINTS TO AN EXISTING BROWNIAN SHEET LATTICE

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Abstract

We compute the conditional distribution of new interior points of a given a lattice representing a path of a Brownian sheet process in discrete time. This is done so that we can simulate paths of this multi-parameter Gaussian process by refining previously simulated paths, which allows one to refine a particular area of the path that is of interest.

Keywords: Brownian sheet; Simulation; Conditional distribution

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1. Introduction

Advances on HJM's model for forward rates using a Brownian sheet process such as Kennedy (1994) and Goldstein (2000) has led to more interest in this two parameter Wiener process. This process gives a more general model of the dynamics of the forward rate curve with the ability to model forward rates that are not perfectly correlated with rates of different expiration times.

It is important to have flexible methods of simulation for this process in order to study interest rate sensitive derivatives using Monte Carlo techniques. Especially in the case of path dependent contingent claims, it is very useful to have a way of refining a part of a given lattice without having to reconstruct a new path. This allows for further refinement of extreme areas of a specific path in order to gain more insight to these sensitive areas.

Currently a Brownian path is constructed by starting at the left side (where the first parameter is zero) and to build from the left to the right by simulating independent normal random variables. When an area of the path requires further refinement,

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this method of simulation does not give us anyway to do that without discarding the previously simulated points.

Paul Levy (1940), introduced a method of simulating a path of Brownian motion by computing the conditional distribution of the Brownian motion at time t_1, given the value of the entire Brownian path before time $t_0 < t_1$ and after time $t_2 > t_1$. Thus, he computed the distribution for B_{t_1} given B_s $\forall$ s $\leq t_0$ and s $\geq t_2$. With this, one could simulate the Brownian motion by simulating the two end points and then adding middle points to refine the approximation to the path. This approach allows for the isolation of a specific time interval of interest and refine the Brownian path only over that interval while still retaining the other points along the simulated path.

In this paper, with the intention of developing a more flexible approach to simulating the Brownian sheet, we would like to follow in the footsteps of Paul Levy and compute the conditional distribution of the Brownian sheet process at a point given the values of the other points on the already existing grid. Like Levy's approach for the one dimensional case, this leads to a method of simulation by refining an already existing grid of points representing the process by adding new points to the interior of the grid while still retaining the original points of the grid. It also gives us a way to refine a particular area of interest on a simulated sheet without having to consider other parts of the sheet.

The Brownian sheet process B_t for $t \in \mathbb{R}_+^2$ is a mean-zero continuous Gaussian random field with covariance

$$Cov(B_{t_1,u_1}, B_{t_2,u_2} = (t_1 \wedge t_2)(u_1 \wedge u_2)$$

Let x and $y \in \Re$. Then for fixed y, $B_{x,y}$ is standard one dimensional Brownian motion. From the definition one can see that on either axis the Brownian sheet vanishes; $B_{x,0} = B_{0,y} = 0$. Two important properties of the Brownian sheet are symmetry which says that

$$B_{x,y} = B_{y,x}$$

and the scaling property

$$B_{ax,bx} = \sqrt{ab}B_{x,y}.$$

2. Computing Conditional Distributions

Let the process p_{n_1,n_2} be the discrete version of the Brownian sheet. Then $\forall$ n_1 and $n_2 \in \{0,\ 1,\ 2,\ \dots\}$ $p_{n_1,n_2} = B_{n_1,n_2}$ Then in this section we show for c $\in$ (0, 1) how to compute the conditional probability

$$P(p(n_1, n_2 + c) < x|p(m_1, m_2)\forall m_1, m_2 \in \{0, 1, 2 \ldots\}).$$

Theorem 2.1. *The increment* $p(n_1+1, n_2+c) - p(n_1, n_2+c)$ *has normal conditional distribution with mean*

$$c[p(n_1 + 1, n_2 + 1) - p(n_1, n_2 + 1)] + (1 - c)[p(n_1 + 1, n_2) - p(n_1, n_2)] \tag{1}$$

and variance

$$c - c^2. \tag{2}$$

Proof. Define the random variable $Y(n_1, n_2)$ as the linear combination of the integer increments of three points on the lattice;

$$\begin{aligned} Y(n_1, n_2) = p(n_1 + 1, n_2 + c) - p(n_1, n_2 + c) \\ - (1 - c)[p(n_1 + 1, n_2) - p(n_1, n_2)] \\ - c[p(n_1 + 1, n_2 + 1) - p(n_1, n_2 + 1)]. \end{aligned} \tag{3}$$

Therefore, $Y(n_1, n_2)$ is distributed normally with mean 0 and covariance structure given by

$$Cov(Y(n_1, n_2), Y(n_1, n_2)) = c - c^2, \forall n_1, n_2 \in \{0, 1, 2, \ldots\}$$

and

$$Cov(Y(n_1, n_2), p(m_1, m_2)) = 0 \forall m_1, m_2 \in \{0, 1, 2, \ldots\}.$$

This implies that $Y(n_1, n_2)$ is independent of the discrete points of our process $p(m_1, m_2)$.

Solving for $p(n_1 + 1, n_2 + c) - p(n_1, n_2 + c)$ we get

$$\begin{aligned} p(n_1 + 1, n_2 + c) - p(n_1, n_2 + c) = \\ Y(n_1, n_2) + c[p(n_1 + 1, n_2 + 1) - p(n_1, n_2 + 1)] + \\ (1 - c)[p(n_1 + 1, n_2) - p(n_1, n_2)]. \end{aligned} \tag{4}$$

If X_1 is a real valued R.V. independent of $\sigma(X_2, \ldots, X_n)$, the σ-field generated by the real valued R.V.s X_2, ..., X_n, then for a continuous function $f \in C(\Re^n, \Re)$

$$P(f(X_1, \ldots, X_n) < y | X_2 = c_2, \ldots, X_n = c_n) = P(f(X_1, c_2, \ldots, c_n) < y). \quad (5)$$

Let the continuous function f be defined by,

$$f(Y(n_1, n_2), p(n_1 + 1, n_2 + 1), p(n_1, n_2 + 1), p(n_1 + 1, n_2), p(n_1, n_2)) = Y(n_1, n_2) + c[p(n_1 + 1, n_2 + 1) - p(n_1, n_2 + 1)] + (1 - c)[p(n_1 + 1, n_2) - p(n_1, n_2)].$$

to get the result.

We can see that the conditional probability of the increments in the x direction of our new points $p(n_1, n_2 + c)$ given all the points at integer coordinates in the grid, depends only on the increments in the x direction of the point directly above and the point directly below it. This will allow us to determine the conditional distribution of a single new points, which allows us to simulate new points anywhere on the grid while keeping the previously simulated points.

Theorem 2.2. *The point $p(n_1, n_2 + c)$, given $p(m_1, m_2)$ $\forall$ m_1, $m_2 \in \{0, 1, 2 \ldots\})$ is distributed normally with mean*

$$cp(n_1, n_2 + 1) + (1 - c)p(n_1, n_2) \quad (6)$$

and variance

$$n_1 c(1 - c) \quad (7)$$

Proof. Consider

$$\begin{aligned} p(n_1, n_2 + c) &= p(n_1, n_2 + c) + p(0, n_2 + c) \\ &= [p(n_1, n_2 + c) - p(n_1 - 1, n_2 + c)] \\ &\quad + p(n_1 - 1, n_2 + c) - \ldots \\ &\quad + [p(1, n_2 + c) - p(0, n_2 + c)] + p(0, n_2 + c). \end{aligned}$$

From the definition of $Y(i, n_2)$ (3)

$$\begin{aligned} p(n_1, n_2 + c) &= \sum_{i=0}^{n_1} Y(i, n_2) + c[p(n_1, n_2 + 1) - p(0, n_2 + 1)] \\ &\qquad + (1 - c)[p(n_1, n_2) - p(0, n_2)] \\ &= \sum_{i=0}^{n_1} Y(i, n_2) + cp(n_1, n_2 + 1) + (1 - c)p(n_1, n_2). \end{aligned}$$

Thus

$$P(p(n_1, n_2 + c) < x | p(m_1, m_2) \forall m_1, m_2 \in \{0, 1, 2 \ldots\}) = \\ P(\sum_{i=0}^{n_1} Y(i, n_2) < x - cp(n_1, n_2 + 1) - (1 - c)p(n_1, n_2)). \quad (8)$$

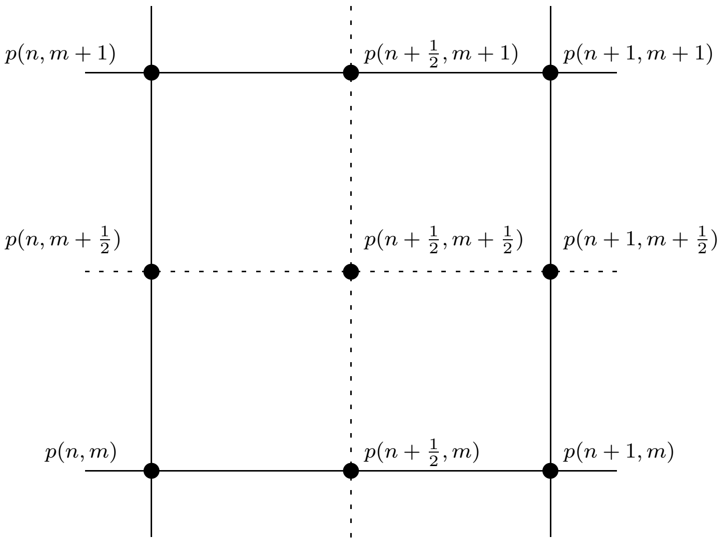
Notice from (3) and (7) that $Y(i, n_2)$ and $Y(j, n_2)$ are independent when $i \neq j \ \forall i, j \in \{0, 1, 2, \ldots\}$. So for a fixed n_2, $Y(i, n_2)$ is a sequence of i.i.d. normal random variables with mean 0 and variance $c - c^2$. This means $\sum_{i=0}^{n_1} Y(i, n_2)$ is a normal random variable with mean 0 and variance $n_1 c(1 - c)$. The result follows.

3. Simulation of the Brownian Sheet Process

We now give an algorithm, which uses the previous results, for the simulation of a Brownian sheet process when taking the special case of $c = \frac{1}{2}$. This is done starting with an existing grid representing the process and showing how to use the special case of the results to refine the grid by reducing the distance between the grid points by half.

We start with an $(N + 1) \times (M + 1)$ lattice representing the Brownian sheet from $(0, 0)$ to (T_x, T_y) with increments of $\frac{1}{N}$ in the x direction and $\frac{1}{M}$ in the y direction. This means that for each n $\in$ {0, 1, …, N} and m $\in$ { 0, 1, …, M} we have a point p(n,m) on the lattice s.t. $p(n, m) = B_{\frac{nT_x}{N}, \frac{mT_y}{M}}$. The goal then is to refine this lattice by decreasing the increment size by half in both the x and in the y direction, thus leaving us with an $(2N + 1) \times (2M + 1)$ lattice with increments of $\frac{1}{2N}$ in the x direction and $\frac{1}{2M}$ in the y direction. This will require that for each n and m we must add the new points p(n, m + $\frac{1}{2}$), p(n + $\frac{1}{2}$, m), p(n + $\frac{1}{2}$, m + $\frac{1}{2}$), p(n + $\frac{1}{2}$, m + 1), and p(n +1, m + $\frac{1}{2}$) to the lattice. This is illustrated below.

FIGURE 1: Lattice Representing the Brownian Sheet



Each of these new points are dependent on the other points of the lattice. We need to simulate them keeping this dependence in mind. To simulate the Brownian sheet from (0,0) to (T_x, T_y) for arbitrary T_x and T_y we need only simulate the Brownian sheet from (0,0) to (1,1) and multiply each simulated point by the constant $\sqrt{T_x T_y}$. Likewise, regardless of the mesh size in the x direction or in the y direction each point p(n,m) on the grid can be represented as $\sqrt{\frac{T_x}{N}\frac{T_y}{M}}B_{n,m}$. Thus, from now on we will consider $p(n, m) = B_{n,m}$.

We proceed by first decreasing the increment size in the y direction only, leaving us with an $(N + 1) \times (2M + 1)$ grid. This turns out to be much simpler to accomplish than trying to decrease the increment size in both directions simultaneously. Then we will decrease the increment size in the x direction of this new grid, which is the same as doing it in the y direction. This will leave us with the desired $(2N + 1) \times (2M + 1)$ grid.

Now we will decrease the increment size in the y direction. For this we must only add the points $p(n, m + \frac{1}{2})$ and $p(n+1, m+\frac{1}{2})$. To generate the point $p(n, m+\frac{1}{2})$ we generate a standard normal random variable Z, and then using Theorem 2, we set

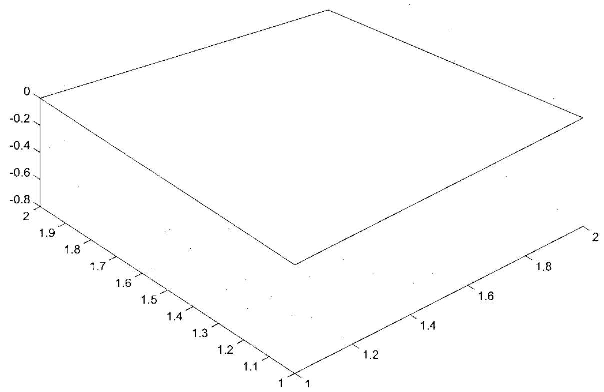
$$p(n, m + \frac{1}{2}) = \frac{\sqrt{n}}{2} Z + \frac{1}{2} p(n, m+1) + \frac{1}{2} p(n, m).$$

Now using Theorem 1 we generate new points starting at $p(n, m + \frac{1}{2})$ by setting $p(n+1, m+\frac{1}{2})$ equal to

$$p(n, m + \frac{1}{2}) + \frac{1}{2}[p(n_1 + 1, n_2 + 1) - p(n_1, n_2 + 1)] + \frac{1}{2}[p(n_1 + 1, n_2) - p(n_1, n_2)] + \frac{1}{2} Z$$

where Z is a newly generated standard normal random variable. Then proceeding by induction and doing the same procedure for every integer m $\in$ {0, 1, ..., M - 1} we have reduced the increment size in the y direction by $\frac{1}{2}$. By doing the same procedure, we decrease the size in the x direction and this leaves us with the refined grid that was our goal.

FIGURE 2: Initial Grid



We then show the refinement of this grid, which gives us a 9 $\times$ 9 grid.

And finally we show the original grid refined 7 times from two different angles. This figure shows the sheet from the top in order to give the reader a view of the 129 $\times$ 129 grid.

The final illustration shows the sheet level with the origin in order to give a more accurate representation of the path of the Brownian sheet.

The original grid was of the form p(0,0) = p(0,1) = p(1,0) = 0 and p(1,1) was distributed as a standard normal random variable.

Figure 3: 3×3

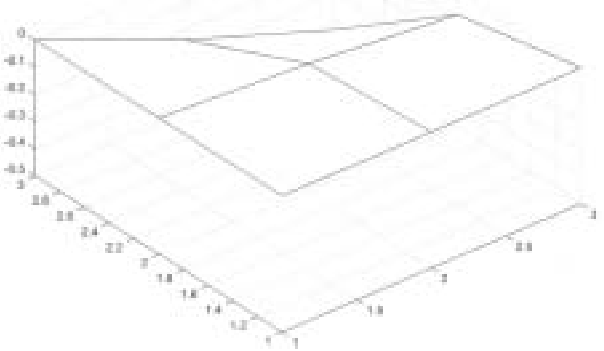
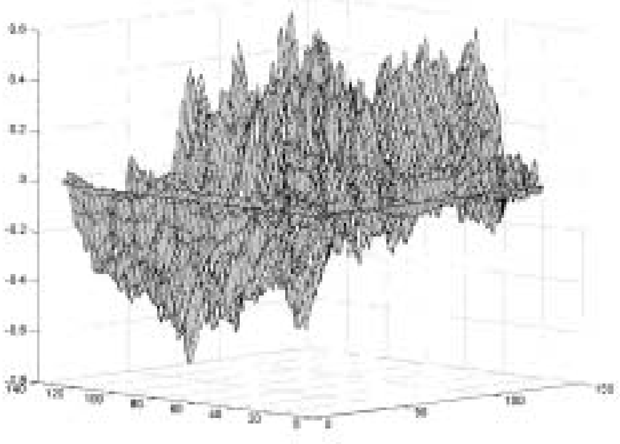


Figure 4: 129×129 Grid



4. Conclusion

In this paper, we have shown that the discrete Brownian sheet process at a point given the values of the other points on the grid has a normal distribution. We then used this result to simulate the Brownian sheet process through the interpolation of points on a given grid. This leads to an alternative way to simulate the process which does not force you to throw out previously simulated points. It also allows a better understanding of the dependence of a point on the grid to the already existing points of the grid; specifically the dependence of the increments.

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References

[1] Feller, W. (1950). *An Introduction to Probability Theory and Its Applications, Volume 1* Third Edition. John Wiley and Sons, New York.

[2] Goldstein, R. (2000). The Term Structure of Interest Rates as a Random Field *The Review of Financial Studies* **13,** 365–384.

[3] Kennedy, D. P. (1994). The Term Structure of Interest Rates as a Gaussian Random Field *Mathematical Financial* **13,** 365–384.

[4] Levy, M. (1940). Le Mouvement Brownian Plan. *American Journal of Mathematics* **62,** 487-550.

[5] Peres, Y. (1998). Path Properties of Brownian Motion. Lecture Notes for Statistics 206B, UC Berkeley.

[6] Santa-Clara, P. and Sornette, D. (1999). The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks. Working Paper, UCLA.

[7] Walsh, J (1986). *Lectures in Probability and Statistics*. Springer-Verlag, Heidelberg Germany.